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On the stability of rotating compressible and inviscid fluids

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Necessary conditions for linear stability of a rotating, compressible and inviscid fluid are found by the generalized progressing wave expansion method. The full three-dimensional problem involving an arbitrarily given rotational symmetric external force field is considered for an arbitrary steady shear flow with vanishing axial velocity. The results obtained are compared with previously known results.

1. Introduction

The stability of rotating fluids has previously been studied in a large number of papers. The fundamental paper on inviscid flows is that of Lord Rayleigh (1916), while viscid flows were first studied by G. I. Taylor (1923). These and a number of later works are discussed in the book by Chandrasekhar (1961). The majority of the previously known results concerns incompressible fluids; only recently some of the problems have been studied for compressible fluids (see Lalas 1975; Warren 1975; Eckhoff & Storesletten 1978, and the references quoted there).

In addition to the small-scale laboratory phenomena such as fluid flows between two coaxial rotating cylinders, the natural background for the study of rotating fluids also include medium-scale geophysical phenomena such as cyclones and large-scale geophysical and astrophysical phenomena such as the basic rotation behaviour of planetary and stellar atmospheres. With this background it seems important to extend the theory to rather general shear flows of compressible fluids subject to a rather general external force field.

In this work we study the stability of general steady azimuthal shear flows of an inviscid compressible fluid subject to a general rotational symmetric external force field. Thus we consider basic states and external force fields which depend both on the distance r from the axis of rotation and on the distance z along this axis. We obtain necessary conditions for linear stability of such basic states with vanishing axial velocity by the same method as we applied to swirling flows in Eckhoff & Storesletten (1978; hereinafter referred to as I) when the basic states and the potential for the external forces were assumed to depend on r only. Our method is based on the generalized progressing wave expansion method (see Friedlander 1958; Ludwig 1960; Eckhoff 1975).

Most of the results known on the problem of stability of shear flows in hydro-

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dynamics have been obtained by the normal mode method, which therefore must be regarded as the conventional approach to this problem. In general this approach leads to an eigenvalue problem for a partial differential operator with boundary conditions. In order to be able to study this eigenvalue problem, assumptions have always been made so far which essentially transfer the problem to an eigenvalue problem for an *ordinary* differential operator. Even with such assumptions the eigenvalue problem is in general very difficult to solve; the stability problems are therefore not completely resolved even in these cases. Recently an attempt to give a common analytical basis for the study of these eigenvalue problems was given by Warren (1976, 1978). In the general case considered in this paper it is not possible to transfer the problem to an eigenvalue problem for an ordinary differential operator, the normal mode approach therefore would not get us very far. As will be seen in the following, however, the generalized progressing wave expansion method gets around these difficulties and enables us to prove necessary conditions for stability of very general shear flows.

When the external force field has *no* axial component, we find that all basic states with axial shear are unstable. Thus a stable basic state is necessarily independent of z in this case; it is therefore a special case of the swirling flows considered by Warren (1975) and I.

When the external force field has an axial component, we find that the picture is radically changed. In order that the basic state shall be stable in this case, the fluid must be stratified in a (statically) stable way and the velocity profile must satisfy a condition which is shown to be a generalization of the classical condition by Lord Rayleigh (1916).

2. The basic equations

The fundamental equations are

$$\left.\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\,.\,\nabla\mathbf{v} = -\rho^{-1}\nabla p+\nabla V,\\ &\frac{\partial \rho}{\partial t}+\mathbf{v}\,.\,\nabla\rho+\rho\nabla\,.\,\mathbf{v} = 0,\\ &\frac{\partial}{\partial t}(p\rho^{-\gamma})+\mathbf{v}\,.\,\nabla(p\rho^{-\gamma}) = 0,\end{aligned}\right\} \tag{2.1}$$

where $\mathbf{v}$ denotes the velocity, ρ the density, p the pressure, V a given potential for the external forces acting on the fluid, and γ is a constant.

In an inertial frame we let (r, ϕ, z) denote cylindrical co-ordinates where the z axis coincides with the axis of symmetry in the basic state. In particular, the potential for the external forces is assumed to be independent of ϕ, i.e. $V = V(r,z)$. The basic flow of the rotating fluid may then be written as

$$\mathbf{v} = v_0(r,z)\,\hat{\boldsymbol{\phi}}, \quad \rho = \rho_0(r,z), \quad p = p_0(r,z). \tag{2.2}$$

Here v_0, ρ_0, p_0 are assumed to satisfy the equations

$$-r^{-1}v_0^2 = -\rho_0^{-1}\frac{\partial p_0}{\partial r}+\frac{\partial V}{\partial r}, \quad 0 = -\rho_0^{-1}\frac{\partial p_0}{\partial z}+\frac{\partial V}{\partial z}. \tag{2.3}$$

In the basic flow (2.2), the rotation as well as the shear are manifested in v_0, which can be chosen arbitrarily. In fact, by eliminating p_0 in (2.3) we get

$$\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right)\frac{\partial \rho_0}{\partial z}-\frac{\partial V}{\partial z}\frac{\partial \rho_0}{\partial r}+2r^{-1}v_0\frac{\partial v_0}{\partial z}\rho_0 = 0. \tag{2.4}$$

For given v_0, (2.4) is a linear partial differential equation of order 1 for ρ_0 which can be solved by standard methods (see Courant & Hilbert 1962). The general solution of (2.4) involves an arbitrary function of one variable. This means that the stratification in the basic flow (2.2) of the fluid may be arbitrarily chosen (in one direction). Finally, when both $v_0(r,z)$ and $\rho_0(r,z)$ have been chosen such that (2.4) is satisfied, $p_0(r,z)$ is determined to within an additive constant by (2.3).

In order to study the stability properties of the basic flow (2.2), we perturb it in a similar way as we did in I, i.e. we introduce the following expressions into (2.1):

$$\left.\begin{aligned}\mathbf{v} &= \rho_0^{-\frac{1}{2}}u_r\hat{\mathbf{r}}+(\rho_0^{-\frac{1}{2}}u_\phi+v_0)\hat{\boldsymbol{\phi}}+\rho_0^{-\frac{1}{2}}u_z\hat{\mathbf{z}},\\ \rho &= \rho_0+c_0^{-1}\rho_0^{\frac{1}{2}}(s_1+s_2),\quad p = p_0+c_0\rho_0^{\frac{1}{2}}s_2.\end{aligned}\right\} \tag{2.5}$$

Here $c_0 = (\gamma p_0/\rho_0)^{\frac{1}{2}}$ denotes the local sound speed. By substituting (2.5) into (2.1), the linearized equations for the perturbations are found to be

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\mathbf{A}^1\frac{\partial \boldsymbol{\omega}}{\partial r}+\mathbf{A}^2\frac{\partial \boldsymbol{\omega}}{\partial \phi}+\mathbf{A}^3\frac{\partial \boldsymbol{\omega}}{\partial z}+\mathbf{B}\boldsymbol{\omega} = 0. \tag{2.6}$$

where $\boldsymbol{\omega} = \{u_r, u_\phi, u_z, s_1, s_2\}$ represents the perturbation superimposed on the basic flow (2.2). Here $\boldsymbol{\omega}$ is treated as a column vector and the coefficient matrices are

$$\mathbf{A}^1 = \begin{pmatrix}0&0&0&0&c_0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\c_0&0&0&0&0\end{pmatrix},\quad \mathbf{A}^2 = \begin{pmatrix}r^{-1}v_0&0&0&0&0\\0&r^{-1}v_0&0&0&r^{-1}c_0\\0&0&r^{-1}v_0&0&0\\0&0&0&r^{-1}v_0&0\\0&r^{-1}c_0&0&0&r^{-1}v_0\end{pmatrix}, \tag{2.7a,b}$$

$$\mathbf{A}^3 = \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&c_0\\0&0&0&0&0\\0&0&c_0&0&0\end{pmatrix},\quad \mathbf{B} = \begin{pmatrix}0&-2r^{-1}v_0&0&\beta_1&G_1\\2r^{-1}v_0-a&0&b&0&0\\0&0&0&\beta_2&G_2\\\alpha_1&0&\alpha_2&0&0\\H_1&0&H_2&0&0\end{pmatrix}, \tag{2.7c,d}$$

where

$$\left.\begin{aligned}&a = r^{-1}v_0-\frac{\partial v_0}{\partial r},\quad b = \frac{\partial v_0}{\partial z},\\ &\alpha_1 = c_0\rho_0^{-1}\frac{\partial \rho_0}{\partial r}-c_0^{-1}\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right),\quad \beta_1 = -c_0^{-1}\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right),\\ &\alpha_2 = c_0\rho_0^{-1}\frac{\partial \rho_0}{\partial z}-c_0^{-1}\frac{\partial V}{\partial z},\quad \beta_2 = -c_0^{-1}\frac{\partial V}{\partial z},\\ &G_1 = c_0^{-1}\left(\frac{\gamma}{2}-1\right)\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right),\quad H_1 = r^{-1}c_0+c_0^{-1}\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right)-\tfrac{1}{2}c_0\rho_0^{-1}\frac{\partial \rho_0}{\partial r},\\ &G_2 = c_0^{-1}\left(\frac{\gamma}{2}-1\right)\frac{\partial V}{\partial z},\quad H_2 = c_0^{-1}\frac{\partial V}{\partial z}-\tfrac{1}{2}c_0\rho_0^{-1}\frac{\partial \rho_0}{\partial z}.\end{aligned}\right\} \tag{2.8}$$

We see that (2.6) is a symmetric hyperbolic system analogous to the system (2.5) in I. The matrix $\mathbf{B}$ obtained here, (2.7*d*), is seen to be more complicated than the corresponding matrix in I, (2.6*d*), while the matrices $\mathbf{A}^j$ obtained here, (2.7*a*, *b*, *c*), are similar to the corresponding matrices in I, (2.6*a*, *b*, *c*), when we put $w_0 = 0$ there. The characteristic equation, the characteristic roots and the eigenvectors associated with these roots are therefore similar here to those in I where we put $w_0 = 0$. In particular, if we let the leading term of the generalized progressing wave solution for the gravity waves (inertial waves) be given by

$$\mathbf{a}_0(r, \phi, z, t) \exp\{i\omega\Phi(r, \phi, z, t)\}, \tag{2.9}$$

where $i = \sqrt{-1}$ and ω denotes the *frequency parameter*, we find that the *phase function* Φ must satisfy

$$\frac{\partial\Phi}{\partial t} + r^{-1} v_0 \frac{\partial\Phi}{\partial\phi} = 0. \tag{2.10}$$

Furthermore, the *amplitude* $\mathbf{a}_0$ must have the form

$$\mathbf{a}_0 = \sigma_1 \mathbf{r}_1 + \sigma_2 \mathbf{r}_2 + \sigma_3 \mathbf{r}_3, \tag{2.11}$$

where σ_1, σ_2, σ_3 are determined by the transport equations which we shall give below, and $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are given by (2.9), (2.14) in I with the quantities ξ^1, ξ^2, ξ^3 defined by

$$\xi^1 = \frac{\partial\Phi}{\partial r}, \quad \xi^2 = \frac{\partial\Phi}{\partial\phi}, \quad \xi^3 = \frac{\partial\Phi}{\partial z}, \tag{2.12}$$

i.e. ξ^1, $r^{-1}\xi^2$, ξ^3 are analogous to the components of the wavenumber vector.

The ray equations for the gravity waves obtained from (2.10) are

$$\left.\begin{aligned} &\frac{dr}{dt} = 0, \quad \frac{d\phi}{dt} = r^{-1} v_0, \quad \frac{dz}{dt} = 0, \\ &\frac{d\xi^1}{dt} = r^{-1}\xi^2 a, \quad \frac{d\xi^2}{dt} = 0, \quad \frac{d\xi^3}{dt} = -r^{-1}\xi^2 b. \end{aligned}\right\} \tag{2.13}$$

The solutions of (2.13) are readily found to be

$$r = r_0, \quad \phi = \phi_0 + r_0^{-1} v_0(r_0, z_0)\, t, \quad z = z_0, \tag{2.14a}$$

$$\xi^1 = \xi^1_0 + r_0^{-1} \xi^2_0 a(r_0, z_0)\, t, \quad \xi^2 = \xi^2_0, \quad \xi^3 = \xi^3_0 - r_0^{-1} \xi^2_0 b(r_0, z_0)\, t, \tag{2.14b}$$

where r_0, ϕ_0, z_0, ξ^1_0, ξ^2_0, ξ^3_0 denote the initial values at $t = 0$.

From (A 15) of I, the transport equations for the gravity waves are found to be in the case considered here

$$\begin{aligned} \frac{d\sigma_1}{dt} = {} & k^{-2}\{-r^{-1}\xi^1\xi^2 a - r^{-1}\xi^2\xi^3(\alpha_1 + \beta_1)\}\,\sigma_1 \\ & + k^{-2}\{-r^{-1}\xi^1\xi^2(2b + \beta_1) + r^{-1}\xi^2\xi^3(2r^{-1}v_0 - a + \alpha_2)\}\,\sigma_2 \\ & + k^{-2}\{(\xi^1)^2 b - \xi^1\xi^3(2r^{-1}v_0 - a + \alpha_2) - (r^{-1}\xi^2)^2(b + \beta_1) + (\xi^3)^2\alpha_1\}\,\sigma_3, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \frac{d\sigma_2}{dt} = {} & k^{-2}\{r^{-1}\xi^1\xi^2(b - \alpha_1) - r^{-1}\xi^2\xi^3(2r^{-1}v_0 - 2a - \beta_2)\}\,\sigma_1 \\ & + k^{-2}\{r^{-1}\xi^1\xi^2(\alpha_2 + \beta_2) + r^{-1}\xi^2\xi^3 b\}\,\sigma_2 \\ & + k^{-2}\{-(\xi^1)^2\alpha_2 - \xi^1\xi^3(b - \alpha_1) + (r^{-1}\xi^2)^2(a + \beta_2) + (\xi^3)^2(2r^{-1}v_0 - a)\}\,\sigma_3, \end{aligned} \tag{2.16}$$

$$\begin{aligned} \frac{d\sigma_3}{dt} = {} & k^{-2}\{\xi^1\xi^3(2r^{-1}v_0 - \beta_2) + (r^{-1}\xi^2)^2(b - \alpha_1) + (\xi^3)^2\beta_1\}\,\sigma_1 \\ & + k^{-2}\{-(\xi^1)^2\beta_2 + \xi^1\xi^3\beta_1 - (r^{-1}\xi^2)^2(a - \alpha_2) - (\xi^3)^2\, 2r^{-1}v_0\}\,\sigma_2 \\ & + k^{-2}\{-r^{-1}\xi^1\xi^2(\alpha_2 + \beta_2) + r^{-1}\xi^2\xi^3(\alpha_1 + \beta_1)\}\,\sigma_3. \end{aligned} \tag{2.17}$$

These transport equations are valid along the rays (2.14). Thus substituting (2.14) into (2.15)–(2.17), we obtain a closed linear system of ordinary differential equations

$$\frac{d\boldsymbol{\sigma}}{dt} = \mathbf{A}(t)\,\boldsymbol{\sigma} \tag{2.18}$$

for the amplitude $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ of the gravity waves. Analogous with I, this is the basic system of equations for our stability analysis.

So far, nothing has been said about boundary conditions. If there are boundaries (e.g. rigid walls or free boundaries), the basic flow (2.2) is of course assumed to satisfy the boundary conditions imposed at these boundaries. Since the rays (2.14*a*) are identical with the streamlines of the basic flow (2.2), they never hit the boundaries if the initial values r_0, ϕ_0, z_0 are restricted to points within the fluid. On these rays the amplitude $\mathbf{a}_0$ in the leading term (2.9) is seen to be uniquely determined by the system of transport equations (2.18) together with the initial values at $t = 0$ of $\boldsymbol{\sigma}$ and ξ_0^1, ξ_0^2, ξ_0^3 (i.e. $\mathbf{a}_0$ is essentially determined by a local analysis). In particular we see from (2.18) that $\mathbf{a}_0 \equiv 0$ in a neighbourhood of the boundaries if the initial values at $t = 0$ of $\boldsymbol{\sigma}$ is chosen to be zero in this neighbourhood. With this choice the leading term (2.9) obviously satisfies the boundary conditions, and these boundary conditions only affect the leading term (2.9) in an arbitrarily small neighbourhood of the boundaries. Within the fluid the leading term (2.9) therefore represents an approximation of a family of solutions of mixed initial-boundary-value problems as well as problems where the fluid is unbounded. As a consequence of this only instabilities which are not sensitive to boundary conditions may be detected in the leading term (2.9).

3. The autonomous case

The system (2.18) is easily seen to be autonomous if and only if

$$\xi_0^2 a(r_0, z_0) = \xi_0^2 b(r_0, z_0) = 0. \tag{3.1}$$

After a considerable amount of algebra, the eigenvalues of the matrix $\mathbf{A}$ in (2.18) are in this case found to be

$$\lambda_1 = 0, \quad \lambda_2 = ik^{-1}D, \quad \lambda_3 = -\lambda_2, \tag{3.2}$$

where D is given by

$$D^2 = -(\xi_0^1)^2\alpha_2\beta_2 + 2\xi_0^1\xi_0^3\alpha_1\beta_2 - (r_0^{-1}\xi_0^2)^2(\alpha_1\beta_1+\alpha_2\beta_2) - (\xi_0^3)^2\{\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0)\}. \tag{3.3}$$

In (3.3) it is assumed that $r = r_0$ and $z = z_0$ have been substituted into α_1, β_1, α_2, β_2, v_0, a.

From standard theory of stability (see Roseau 1966) we conclude that a necessary condition for stability of the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) when (3.1) is satisfied, is that $D^2 \geqslant 0$. Since D^2 is a quadratic form with respect to ξ_0^1, $r^{-1}\xi_0^2$, ξ_0^3, it can be transformed to a diagonal form

$$D^2 = \kappa_1 x^2 + \kappa_2 y^2 - (\alpha_1\beta_1 + \alpha_2\beta_2)(r_0^{-1}\xi_0^2)^2 \tag{3.4}$$

by an orthogonal transformation $(\xi_0^1, \xi_0^3) \to (x, y)$. The coefficients κ_1, κ_2 are then the eigenvalues of the symmetric matrix associated with the ξ_0^1, ξ_0^3 part of the quadratic form (3.3). They are found to be

$$\begin{aligned}\kappa_n = & -\tfrac{1}{2}(\alpha_1\beta_1+\alpha_2\beta_2) - r_0^{-1}v_0(a-2r_0^{-1}v_0) \\ & +(-1)^n\{[\tfrac{1}{2}(\alpha_1\beta_1+\alpha_2\beta_2)+r_0^{-1}v_0(a-2r_0^{-1}v_0)]^2 \\ & +(\alpha_1\beta_2)^2-\alpha_2\beta_2[\alpha_1\beta_1+2r_0^{-1}v_0(a-2r_0^{-1}v_0)]\}^{\frac{1}{2}} \quad (n=1,2). \qquad (3.5)\end{aligned}$$

In order that $D^2 \geq 0$ at a point r_0, ϕ_0, z_0 for every choice of ξ_0^1, ξ_0^3 when $\xi_0^2 = 0$, it is obviously necessary and sufficient that $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ at that point. It is easily seen that $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ if and only if

$$\begin{aligned}&\alpha_1\beta_1+\alpha_2\beta_2+2r_0^{-1}v_0(a-2r_0^{-1}v_0) \leq 0, \\ &\alpha_2\beta_2\{\alpha_1\beta_1+2r_0^{-1}v_0(a-2r_0^{-1}v_0)\} \geq (\alpha_1\beta_2)^2. \qquad (3.6)\end{aligned}$$

From this we may conclude the following.

Lemma 1. In order that the trivial solution of (2.18) shall be stable for all ξ_0^1, ξ_0^2, ξ_0^3 at a given point r_0, ϕ_0, z_0, it is necessary that

$$\alpha_1\beta_1+2r_0^{-1}v_0(a-2r_0^{-1}v_0) \leq 0, \qquad (3.7)$$

$$\alpha_2\beta_2 \leq 0, \qquad (3.8)$$

$$\alpha_2\beta_2\{\alpha_1\beta_1+2r_0^{-1}v_0(a-2r_0^{-1}v_0)\} \geq (\alpha_1\beta_2)^2, \qquad (3.9)$$

hold at that point.

From standard theory of stability we conclude that $D^2 > 0$ is a sufficient condition to ensure stability of the trivial solution of (2.18) when $\xi_0^2 = 0$. Hence the trivial solution of (2.18) is always stable when $\xi_0^2 = 0$ and the strict inequalities hold in (3.7)–(3.9). If, on the other hand, equality holds in one of these inequalities, the eigenvalues (3.2) are not necessarily simple any longer. In these marginal cases a more detailed analysis is therefore needed in order to settle the stability problem for (2.18). This will be done in §5.

At points r_0, ϕ_0, z_0 where $a = b = 0$, if any, (3.1) is satisfied for every choice of $\xi_0^1, \xi_0^2, \xi_0^3$. Thus the trivial solution of (2.18) will be unstable for some choices of $\xi_0^1, \xi_0^2, \xi_0^3$ at those points unless

$$\alpha_1\beta_1+\alpha_2\beta_2 \leq 0 \qquad (3.10)$$

holds there in addition to (3.7)–(3.9). However, (3.10) is necessarily satisfied when the strict inequality holds in (3.8). In fact, (3.8) then implies that α_2 and β_2 have opposite signs, and since $b = 0$ it follows from (2.4) that $\alpha_1\beta_2 = \alpha_2\beta_1$. Thus either $\alpha_1 = \beta_1 = 0$ or α_1 and β_1 have opposite signs, consequently $\alpha_1\beta_1 \leq 0$. The marginal case where $\alpha_2\beta_2 = 0$ will be discussed in §5.

4. The non-autonomous case

We now consider the cases where (3.1) is *not* satisfied, i.e. we assume that

$$m^2 \stackrel{\text{def}}{=} a^2(r_0, z_0)+b^2(r_0, z_0) \neq 0, \quad \xi_0^2 \neq 0. \qquad (4.1)$$

It is easily seen in these cases that the matrix $\mathbf{A}(t)$ in (2.18) is analytic at $t = +\infty$, i.e. that for $t > t_0 > 0$, $A(t)$ can be written as a convergent power series

$$\mathbf{A}(t) = \sum_{k=0}^{\infty}\left(\frac{1}{t}\right)^k \mathbf{A}_k. \qquad (4.2)$$

The first two matrices in this expansion are found to be

$$\mathbf{A}_0 = \begin{Bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ a_3 & a_4 & 0 \end{Bmatrix}, \quad \mathbf{A}_1 = \begin{Bmatrix} d_1 & d_2 & b_1 \\ d_3 & d_4 & b_2 \\ b_3 & b_4 & d_5 \end{Bmatrix}, \tag{4.3}$$

where

$$\left.\begin{aligned} a_1 &= m^{-2}(2abr_0^{-1}v_0 + ab\alpha_2 + b^2\alpha_1), \quad a_2 = m^{-2}(2b^2r_0^{-1}v_0 - ab\alpha_1 - a^2\alpha_2), \\ a_3 &= m^{-2}(-2abr_0^{-1}v_0 + ab\beta_2 + b^2\beta_1), \quad a_4 = m^{-2}(-2b^2r_0^{-1}v_0 - ab\beta_1 - a^2\beta_2); \end{aligned}\right\} \tag{4.4}$$

$$\left.\begin{aligned} d_1 &= m^{-2}(b\alpha_1 + b\beta_1 - a^2), \quad d_2 = m^{-2}(-a\beta_1 - 2br_0^{-1}v_0 - ba - b\alpha_2), \\ d_3 &= m^{-2}(-a\alpha_1 + 2br_0^{-1}v_0 - ba - b\beta_2), \quad d_4 = m^{-2}(a\alpha_2 + a\beta_2 - b^2), \\ d_5 &= m^{-2}(-a\alpha_2 - a\beta_2 - b\alpha_1 - b\beta_1); \end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned} b_1 &= (r_0^{-1}\xi_0^2 m^2)^{-1}\{\xi_0^1(-2aa_1 + 2br_0^{-1}v_0 + ab + b\alpha_2) \\ &\quad + \xi_0^3(2ba_1 - 2ar_0^{-1}v_0 + a^2 - a\alpha_2 - 2b\alpha_1)\}, \\ b_2 &= (r_0^{-1}\xi_0^2 m^2)^{-1}\{\xi_0^1(-2aa_2 + b^2 - 2a\alpha_2 - b\alpha_1) \\ &\quad + \xi_0^3(2ba_2 - 4br_0^{-1}v_0 + ab + a\alpha_1)\}, \\ b_3 &= (r_0^{-1}\xi_0^2 m^2)^{-1}\{\xi_0^1(-2aa_3 - 2br_0^{-1}v_0 + b\beta_2) \\ &\quad + \xi_0^3(2ba_3 + 2ar_0^{-1}v_0 - a\beta_2 - 2b\beta_1)\}, \\ b_4 &= (r_0^{-1}\xi_0^2 m^2)^{-1}\{\xi_0^1(-2aa_4 - 2a\beta_2 - b\beta_1) \\ &\quad + \xi_0^3(2ba_4 + 4br_0^{-1}v_0 + a\beta_1)\}. \end{aligned}\right\} \tag{4.6}$$

From (4.3) it follows that

$$\det\left\{\mathbf{A}_0 + \frac{1}{t}\mathbf{A}_1 - \lambda\mathbf{I}\right\} = -\lambda^3 - \frac{1}{t}\theta\lambda^2 - \left\{\omega + \frac{1}{t}\mu + O\left(\frac{1}{t^2}\right)\right\}\lambda - \frac{1}{t}\nu + O\left(\frac{1}{t^2}\right), \tag{4.7}$$

where

$$\left.\begin{aligned} \theta &= -d_1 - d_4 - d_5, \quad \omega = -a_1a_3 - a_2a_4, \\ \mu &= -a_1b_3 - a_2b_4 - a_3b_1 - a_4b_2, \\ \nu &= a_1a_3d_4 - a_1a_4d_3 + a_2a_4d_1 - a_2a_3d_2. \end{aligned}\right\} \tag{4.8}$$

Using (2.4) and (4.4)–(4.6), the expressions (4.8) may be written as

$$\theta = 1, \quad \omega = \nu = \frac{M^2}{m^2}, \quad \mu = 2\frac{(\xi_0^1 b + \xi_0^3 a)S}{r_0^{-1}\xi_0^2 m^2}, \tag{4.9}$$

where

$$M^2 = -\alpha_2\beta_2a^2 - 2\alpha_1\beta_2ab - \{\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0)\}b^2, \tag{4.10}$$

$$S = \{\alpha_1\beta_1 - \alpha_2\beta_2 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0)\}ab - \alpha_1\beta_2(a^2 + b^2). \tag{4.11}$$

From (4.7), (4.9) the eigenvalues of $\mathbf{A}_0$ are found to be

$$\lambda_1 = 0, \quad \lambda_2 = i\frac{M}{m}, \quad \lambda_3 = -\lambda_2. \tag{4.12}$$

From standard theory of stability we conclude that a necessary condition for stability of the trivial solution of (2.18) is that $M^2 \geqslant 0$. However, if we look at the expression (4.10) as a symmetric quadratic form with respect to a, b, it can be transformed to a diagonal form which is easily found to be

$$M^2 = \kappa_1 K^2 + \kappa_2 L^2, \tag{4.13}$$

where κ_1, κ_2 are given by (3.5) and K, L are two linearly independent combinations of a and b. Thus $M^2 \geqslant 0$ when the conditions in lemma 1 are satisfied. Furthermore, (4.13) shows that $M = 0$ is only possible when equality holds in at least one of the inequalities (3.7)–(3.9). As we have noted earlier, these marginal cases will be studied in § 5.

Let us now assume that the strict inequalities hold in (3.7)–(3.9). The eigenvalues (4.12) of $\mathbf{A}_0$ are then distinct; we may therefore consider the following asymptotic equations as $t \to +\infty$

$$\det\left\{\mathbf{A}_0 + \frac{1}{t}\mathbf{A}_1 - \left(\lambda_j + \frac{1}{t}r_j\right)\mathbf{I}\right\} = O\left(\frac{1}{t^2}\right); \quad j = 1, 2, 3. \tag{4.14}$$

Using (4.7), (4.9) and (4.12) the solutions are found to be

$$r_1 = -1, \quad r_2 = i\frac{1}{r_0^{-1}\xi_0^2 m^3}(\xi_0^1 b + \xi_0^3 a)\frac{S}{M}, \quad r_3 = -r_2. \tag{4.15}$$

The theorems 2.1 and 4.1 in Coddington & Levinson (1955, cha. 5), imply that there exist linearly independent solutions $\boldsymbol{\sigma}_{(1)}(t)$, $\boldsymbol{\sigma}_{(2)}(t)$, $\boldsymbol{\sigma}_{(3)}(t)$ of (2.18), such that for some constant vectors $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ we have asymptotically as $t \to +\infty$

$$\|\boldsymbol{\sigma}_{(j)}(t) - \mathbf{P}_j \exp(\lambda_j t + r_j \ln t)\| = O(t^{\mathrm{Re}\, r_j - 1} e^{\mathrm{Re}\,\lambda_j t}). \tag{4.16}$$

Thus a sufficient condition for stability of the trivial solution of (2.18) is that the three expressions $\exp(\lambda_j t + r_j \ln t)$, $j = 1, 2, 3$, are bounded as $t \to +\infty$. From (4.12), (4.15) it is easily seen that this is always the case. In view of the discussion at the end of the preceding section, we have therefore proved

Lemma 2. Assume that

$$\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0) < 0, \tag{4.17}$$

$$\alpha_2\beta_2 < 0, \tag{4.18}$$

$$\alpha_2\beta_2\{\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0)\} > (\alpha_1\beta_2)^2, \tag{4.19}$$

are satisfied at a given point r_0, ϕ_0, z_0. The trivial solution of (2.18) is then stable for all ξ_0^1, ξ_0^2, ξ_0^3 at this point.

5. The marginal cases

In this section we shall study the stability properties of the trivial solution of (2.18) in all cases which are intermediate to the lemmas 1 and 2, i.e. we shall study the cases where equality holds in at least one of the inequalities (3.7)–(3.9).

Lemma 3. Assume that $\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0) = \alpha_2\beta_2 = 0$ at a given point r_0, ϕ_0, z_0. The trivial solution of (2.18) is then stable for all ξ_0^1, ξ_0^2, ξ_0^3 if and only if

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = v_0 = a = b = 0 \tag{5.1}$$

at the point r_0, ϕ_0, z_0.

Proof. Suppose that the trivial solution of (2.18) is stable for all ξ_0^1, ξ_0^2, ξ_0^3. Then $\alpha_1\beta_2 = 0$ by the inequality (3.9) in lemma 1. If we choose $\xi_0^2 = 0$, we therefore obtain $\lambda_1 = \lambda_2 = \lambda_3 = 0$ from (3.2), (3.3), which implies that the matrix $\mathbf{A}$ in (2.18) must vanish when $\xi_0^2 = 0$. This is easily seen to imply (5.1). If on the other hand we suppose that (5.1) is satisfied, the trivial solution of (2.18) is obviously stable for all ξ_0^1, ξ_0^2, ξ_0^3.

Lemma 4. Assume that $\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0) = 0$ and $\alpha_2\beta_2 < 0$ at a given point r_0, ϕ_0, z_0. The trivial solution of (2.18) is then stable for all $\xi_0^1, \xi_0^2, \xi_0^3$ if and only if

$$\alpha_1 = \beta_1 = v_0 = a = 0 \tag{5.2}$$

at the point r_0, ϕ_0, z_0.

Proof. Suppose that the trivial solution of (2.18) is stable for all $\xi_0^1, \xi_0^2, \xi_0^3$. If we choose $\xi_0^1 = \xi_0^2 = 0$, we see from (3.2), (3.3) that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This implies that the matrix $\mathbf{A}$ in (2.18) must vanish when $\xi_0^1 = \xi_0^2 = 0$, which is easily seen to imply (5.2). On the other hand, suppose now that (5.2) is satisfied. In the autonomous case, i.e. when (3.1) is satisfied, the trivial solution of (2.18) is then easily seen to be stable. In the non-autonomous case, i.e. when $\xi_0^2 b \neq 0$, it follows from (4.3), (4.4) that $\mathbf{A}_0 = 0$. By introducing $\tau = \ln t$ as a new independent variable in (2.18), we obtain a system where the trivial solution has exactly the same stability properties as the trivial solution of (2.18). Asymptotically as $\tau \to +\infty$ this transformed system tends to a system with $\mathbf{A}_1$ given by (4.3), (4.5) and (4.6) as coefficient matrix. From standard theory of stability it is not difficult to show that in this case the trivial solution of (2.18) is stable if and only if the trivial solution of

$$\frac{d\boldsymbol{\sigma}}{d\tau} = \mathbf{A}_1 \boldsymbol{\sigma} \tag{5.3}$$

is stable. The latter follows since the eigenvalues of $\mathbf{A}_1$ are found to be

$$\lambda_1 = 0, \quad \lambda_2 = -\tfrac{1}{2} + (\tfrac{1}{4} - R_1)^{\frac{1}{2}}, \quad \lambda_3 = -\tfrac{1}{2} - (\tfrac{1}{4} - R_1)^{\frac{1}{2}}, \tag{5.4}$$

where
$$R_1 = -\left\{1 + \left(\frac{\xi_0^1}{r_0^{-1}\xi_0^2}\right)^2\right\}\frac{\alpha_2\beta_2}{b^2} \tag{5.5}$$

is seen to be positive by the assumptions in the lemma.

Lemma 5. Assume that $\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0) < 0$ and $\alpha_2\beta_2 = 0$ at a given point r_0, ϕ_0, z_0. The trivial solution of (2.18) is then stable for all $\xi_0^1, \xi_0^2, \xi_0^3$ if and only if

$$\alpha_2 = \beta_2 = b = 0, \tag{5.6}$$

$$\alpha_1\beta_1 \leqslant 0, \tag{5.7}$$

and furthermore at least *one* of the following conditions

(i) $a \neq 0$, (ii) $\alpha_1 = \beta_1 = 0$, (iii) $\alpha_1\beta_1 < 0$

are satisfied at the point r_0, ϕ_0, z_0.

Proof. Suppose that the trivial solution of (2.18) is stable for all $\xi_0^1, \xi_0^2, \xi_0^3$. If we choose $\xi_0^2 = \xi_0^3 = 0$, we see from (3.2), (3.3) that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This implies that the matrix $\mathbf{A}$ in (2.18) must vanish when $\xi_0^2 = \xi_0^3 = 0$, which is easily seen to imply (5.6). If we assume $a = 0$ also, (3.10) implies (5.7). If $a \neq 0$, we consider the non-autonomous case, i.e. the case where $\xi_0^2 \neq 0$. It then follows from (4.3), (4.4) that $\mathbf{A}_0 = 0$ and as in the proof of the preceding lemma the stability properties are therefore determined by the matrix $\mathbf{A}_1$ given by (4.3), (4.5) and (4.6). The eigenvalues of $\mathbf{A}_1$ are found to be

$$\lambda_1 = 0, \quad \lambda_2 = -\tfrac{1}{2} + (\tfrac{1}{4} - R_2)^{\frac{1}{2}}, \quad \lambda_3 = -\tfrac{1}{2} - (\tfrac{1}{4} - R_2)^{\frac{1}{2}}, \tag{5.8}$$

where
$$R_2 = -\frac{\alpha_1\beta_1}{a^2} - \left(\frac{\xi_0^3}{r_0^{-1}\xi_0^2}\right)^2 \frac{1}{a^2}\{\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0)\}. \tag{5.9}$$

In order that the trivial solution of (5.3) shall be stable in this case, it is therefore necessary that $R_2 \geqslant 0$ for all $\xi_0^1, \xi_0^2, \xi_0^3$ such that $\xi_0^2 \neq 0$. By choosing $\xi_0^3 = 0$ in (5.9), the necessity of (5.7) is established. It now remains to prove the necessity of (ii) when $a = \alpha_1\beta_1 = 0$. If we in this case choose $\xi_0^1 = \xi_0^3 = 0$, we see from (3.2), (3.3) that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This implies that the matrix $\mathbf{A}$ in (2.18) must vanish, which is easily seen to imply (ii). On the other hand, in view of the above considerations it is not difficult to show that the trivial solution of (2.18) is stable for all $\xi_0^1, \xi_0^2, \xi_0^3$ if (5.6), (5.7) and at least one of the conditions (i), (ii), (iii) are satisfied.

Lemma 6. Assume that $\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0) < 0$, $\alpha_2\beta_2 < 0$ and

$$\alpha_2\beta_2\{\alpha_1\beta_1 + 2r_0^{-1}v_0(a - 2r_0^{-1}v_0)\} = (\alpha_1\beta_2)^2$$

at a given point r_0, ϕ_0, z_0. The trivial solution of (2.18) is then stable for all $\xi_0^1, \xi_0^2, \xi_0^3$ if and only if

$$\alpha_1 b + \alpha_2 a = v_0 = 0 \tag{5.10}$$

at the point r_0, ϕ_0, z_0.

Proof. Suppose that the trivial solution of (2.18) is stable for all $\xi_0^1, \xi_0^2, \xi_0^3$. If we choose $\xi_0^1 = (\alpha_1/\alpha_2)\xi_0^3$ and $\xi_0^2 = 0$, we see from (3.2), (3.3) that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This implies that the matrix $\mathbf{A}$ in (2.18) must vanish, which is easily seen to imply (5.10). On the other hand, suppose that (5.10) is satisfied. In the autonomous case, i.e. when (3.1) is satisfied, it is then easily seen that the trivial solution of (2.18) is stable. In the non-autonomous case, i.e. when (4.1) is satisfied, it follows from (4.3), (4.4) that $\mathbf{A}_0 = 0$. As in the proofs of the preceding two lemmas, the stability properties are therefore determined by the matrix $\mathbf{A}_1$ given by (4.3), (4.5) and (4.6). The eigenvalues of $\mathbf{A}_1$ are found to be

$$\lambda_1 = 0, \quad \lambda_2 = -\tfrac{1}{2} + (\tfrac{1}{4} - R_3)^{\frac{1}{2}}, \quad \lambda_3 = -\tfrac{1}{2} - (\tfrac{1}{4} - R_3)^{\frac{1}{2}}, \tag{5.11}$$

where

$$R_3 = -\frac{1}{a^2+b^2}\left\{1 + \frac{(\xi_0^1 b + \xi_0^3 a)^2}{(r_0^{-1}\xi_0^2)^2(a^2+b^2)}\right\}(\alpha_1\beta_1 + \alpha_2\beta_2). \tag{5.12}$$

From (5.10) and the assumptions in the lemma it follows that $R_3 > 0$; thus (5.11) shows that the trivial solution of (5.3) is stable. Consequently the trivial solution of (2.18) is stable.

6. Discussion of stability

At an arbitrarily given initial point r_0, ϕ_0, z_0 in the fluid, the six lemmas proved in the preceding three sections give necessary and sufficient conditions for the trivial solution of (2.18) to be stable for all $\xi_0^1, \xi_0^2, \xi_0^3$ in all possible cases. As far as it is possible to draw conclusions from the leading term in the generalized progressing wave expansion for the gravity waves, it is therefore possible from these lemmas to draw the optimal obtainable necessary conditions for the basic flow (2.2) to be stable (see Eckhoff 1975).

We first consider the special case where V is independent of z, i.e. $\beta_2 \equiv 0$ and consequently this case is *always* marginal. From the lemmas 1, 3 and 5 we see that the basic flow (2.2) cannot be stable unless

$$\alpha_2 = b \equiv 0, \tag{6.1}$$

i.e. it is necessary that the basic flow (2.2) is independent of z. In order to obtain further necessary conditions from the lemmas 1, 3 and 5, we may therefore assume the basic flow (2.2) to be independent of z. However, the problem is then a special case of the problem discussed in I; it is therefore unnecessary to repeat the details here. We sum up the results thus.

Theorem 1. Let the potential for the external forces V be independent of z. In order that the basic flow (2.2) shall be stable it is necessary that the basic flow (2.2) is independent of z and that

$$\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right)\left\{\rho_0^{-1}\frac{\partial \rho_0}{\partial r}-c_0^{-2}\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right)\right\} \geqslant \begin{cases} -2r^{-1}v_0\left(r^{-1}v_0+\dfrac{\partial v_0}{\partial r}\right) & \text{when} \quad r^{-1}v_0\left(r^{-1}v_0+\dfrac{\partial v_0}{\partial r}\right)<0, \\ 0 & \text{when} \quad r^{-1}v_0\left(r^{-1}v_0+\dfrac{\partial v_0}{\partial r}\right)\geqslant 0 \end{cases} \tag{6.2}$$

holds everywhere in the fluid. If equality holds in (6.2) on some set of positive measure, it is furthermore necessary for stability of the flow (2.2) that

$$v_0=\frac{\partial v_0}{\partial r}=\frac{\partial \rho_0}{\partial r}=\frac{\partial V}{\partial r}=0 \tag{6.3}$$

holds almost everywhere on this set.

We now consider the cases where $\partial V/\partial z \neq 0$ almost everywhere in the fluid, i.e. we assume that $\beta_2 \neq 0$ almost everywhere. From the lemmas 1, 3 and 5 we then see that the basic flow (2.2) cannot be stable unless

$$\alpha_2\beta_2<0 \tag{6.4}$$

holds almost everywhere in the fluid. This condition together with lemma 1 gives that it is also necessary for stability of the flow (2.2) that

$$\alpha_1\beta_1+2r^{-1}v_0(a-2r^{-1}v_0)\leqslant\frac{\beta_2}{\alpha_2}(\alpha_1)^2 \tag{6.5}$$

holds almost everywhere in the fluid. We note that all the conditions in lemma 1 are satisfied when (6.4), (6.5) are satisfied. Furthermore, using (2.4) we may rewrite (6.5) in the following way

$$2r^{-1}v_0(a-2r^{-1}v_0)\leqslant-\frac{\alpha_1}{\alpha_2}2r^{-1}v_0\,b. \tag{6.6}$$

If equality holds in (6.5) (or equivalently in (6.6)) on some set of positive measure, it follows from lemma 6 that the basic flow (2.2) cannot be stable unless

$$v_0=0 \tag{6.7}$$

almost everywhere on this set, which implies that $a=b=0$ almost everywhere on this set also. Finally, lemma 4 shows that if the equality holds in (6.5) (or equivalently in (6.6)) and $\alpha_1=0$ on some set of positive measure, then the basic flow (2.2) cannot be stable unless

$$\beta_1=0 \tag{6.8}$$

almost everywhere on this set. When $v_0=\alpha_1=0$, however, (6.8) is automatically satisfied by (2.4). Summing up, we have proved therefore the following theorem.

Theorem 2. Let the potential for the external forces V be such that $\partial V/\partial z \neq 0$ almost everywhere. In order that the basic flow (2.2) shall be stable it is necessary that

$$\rho_0^{-1}\frac{\partial \rho_0}{\partial z}\frac{\partial V}{\partial z} - c_0^{-2}\left(\frac{\partial V}{\partial z}\right)^2 > 0, \tag{6.9}$$

$$r^{-1}v_0\left(r^{-1}v_0+\frac{\partial v_0}{\partial r}\right) \geqslant \frac{\rho_0^{-1}\dfrac{\partial \rho_0}{\partial r} - c_0^{-2}\left(r^{-1}v_0^2+\dfrac{\partial V}{\partial r}\right)}{\rho_0^{-1}\dfrac{\partial \rho_0}{\partial z} - c_0^{-2}\dfrac{\partial V}{\partial z}}\, r^{-1}v_0\frac{\partial v_0}{\partial z}, \tag{6.10}$$

hold almost everywhere in the fluid. If equality holds in (6.10) on some set of positive measure, it is furthermore necessary for stability of the flow (2.2) that

$$v_0 = 0 \tag{6.11}$$

almost everywhere on this set.

7. Interpretation of the results

When the potential for the external forces V is independent of z, theorem 1 shows that the basic flow (2.2) is unstable if v_0 depends on z, i.e. if there is *shear* in the direction orthogonal to the external forces. It is possible, however, to show that this instability is rather weak, i.e. the perturbations have a linear growth. When v_0 is independent of z, ρ_0 and p_0 are necessarily independent of z by (2.3) in this case. Thus it only remains to discuss the conditions (6.2), (6.3). This is partly done in I.

When the potential for the external forces is such that $\partial V/\partial z \neq 0$ almost everywhere in the fluid, condition (6.9) in theorem 2 together with the equilibrium equation (2.4) show that, in the *static* case where $v_0 \equiv 0$, it is necessary for stability that the fluid is *stratified* with $\nabla\rho_0$ parallel to ∇V. The stratification must be such that the local Brunt–Väisälä frequency N is a positive real number almost everywhere. In fact, here we define

$$N^2 = N_z^2 + N_r^2, \tag{7.1}$$

where

$$N_z^2 = \rho_0^{-1}\frac{\partial \rho_0}{\partial z}\frac{\partial V}{\partial z} - c_0^{-2}\left(\frac{\partial V}{\partial z}\right)^2,$$

$$N_r^2 = \rho_0^{-1}\frac{\partial \rho_0}{\partial r}\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right) - c_0^{-2}\left(r^{-1}v_0^2+\frac{\partial V}{\partial r}\right)^2 \tag{7.2}$$

(see Eckart 1960; I). Since $v_0 \equiv 0$ here, it follows from (6.9) and (2.4) that $N_z^2 > 0$ and $N_r^2 \geqslant 0$ almost everywhere.

We now consider the non-static cases when $\partial V/\partial z \neq 0$ almost everywhere. Since the leading term in the generalized progressing wave expansion for the gravity waves is not seriously affected by the presence of boundaries (see the discussion at the end of § 2), we may consider the set of points where $v_0 = 0$ and the set of points where $v_0 \neq 0$ separately. Since we have discussed the static case already, it therefore suffices to consider the cases where $v_0 \neq 0$ almost everywhere. In order that the basic flow (2.2) shall be stable in these cases, condition (6.9) in theorem 2 shows that the fluid must be stratified in a (statically) stable way in the axial direction. Furthermore, theorem 2 shows that it is also necessary for stability that

$$r^{-1}v_0\left(r^{-1}v_0+\frac{\partial v_0}{\partial r}\right) > \frac{\rho_0^{-1}(\partial\rho_0/\partial r)-c_0^{-2}(r^{-1}v_0^2+\partial V/\partial r)}{\rho_0^{-1}(\partial\rho_0/\partial z)-c_0^{-2}(\partial V/\partial z)}\, r^{-1}v_0\frac{\partial v_0}{\partial z}$$

$$= \frac{\alpha_1}{\alpha_2}r^{-1}v_0\, b \tag{7.3}$$

holds almost everywhere in the fluid.

If we compare these results with those we obtained in the cases covered by theorem 1, we find that the picture is radically changed. On the one hand, the presence of an axial component in the external force field implies that there may exist stable flows with axial shear. On the other hand, when v_0 is independent of z, we got (6.2) as a necessary condition for stability when $\partial V/\partial z \equiv 0$, while from (7.3) we get

$$\tfrac{1}{2}r^{-3}\frac{\partial}{\partial r}(rv_0)^2 = r^{-1}v_0\left(r^{-1}v_0+\frac{\partial v_0}{\partial r}\right) > 0 \tag{7.4}$$

as a necessary condition for stability when $\partial V/\partial z \neq 0$ almost everywhere. Thus there is a strict limit on the radial shear in a stable basic flow (2.2) when $\partial V/\partial z \neq 0$, while a sufficiently strong stratification can stabilize any radial shear when $\partial V/\partial z \equiv 0$ (see Warren 1975; I).

The above results are directly applicable in the classical problem of a flow between two infinitely long coaxial rotating cylinders. If the cylinders are placed vertically in a gravitational field, we have proved that the velocity has to satisfy (7.4) and that the fluid must be stratified in a (statically) stable way in the axial direction. These conditions must be satisfied independently of how weak or strong the gravitational field actually is. Since the criteria for stability obtained in the case $V \equiv 0$ are radically different from those obtained when $\partial V/\partial z \neq 0$, it is never possible to neglect the effect of gravity in this problem. It is interesting to note that (7.4) is exactly the classical criterion obtained by Lord Rayleigh (1916) for the stability of the flow of an inviscid incompressible and homogeneous fluid between two coaxial rotating cylinders. The reasoning applied by Lord Rayleigh gives a physical interpretation of the condition (7.4) (see Chandrasekhar 1961, p. 273); thus we have a certain understanding of the physical mechanism behind the general condition (7.3) also.

Since the majority of the previously known results on rotating fluids concerns incompressible and homogeneous fluids, it may be appropriate to consider the corresponding limit in the above criteria when v_0 is independent of z, i.e. the limit where $c_0 \to \infty$, $\partial\rho_0/\partial r \to 0$ and $\partial\rho_0/\partial z \to 0$. Since Lord Rayleigh's criterion (7.4) is independent of these quantities, it must be valid in this limit when $v_0 \neq 0$ and $\partial V/\partial z \neq 0$ almost everywhere. When $\partial V/\partial z \equiv 0$ and $v_0 \neq 0$ almost everywhere we have shown that (6.2) with strict inequality is the relevant stability criterion instead of (7.4). Let $\epsilon > 0$ be an arbitrarily given small number. By assuming that the compressibility and the inhomogeneity of the fluid are sufficiently small (depending on ϵ) it then follows from (7.2) that

$$|N_r^2| < \epsilon \tag{7.5}$$

($N_z \equiv 0$ in this case). From theorem 1, (6.2) it now follows that the basic flow (2.2) is unstable unless the following inequalities

$$0 < N_r^2 < \epsilon, \tag{7.6}$$

$$-r^{-3}\frac{\partial}{\partial r}(rv_0)^2 = -2r^{-1}v_0\left(r^{-1}v_0+\frac{\partial v_0}{\partial r}\right) < \epsilon \tag{7.7}$$

are satisfied almost everywhere in the fluid. In the limit $\epsilon \to 0$ (7.7) is exactly Lord Rayleigh's criterion (7.4), which therefore is the relevant criterion for an incompressible and homogeneous fluid when there is no external force field also, in accordance with Lord Rayleigh's original work (1916). We note that in order to retain stability of the basic flow (2.2) in the above limit, we have to allow a small amount of compressibility and inhomogeneity corresponding to the inequality (7.6). In fact, if $N_r \equiv 0$ the basic flow (2.2) will be unstable and the perturbations will have a linear growth. This is a marginal case which is analogous to the case $N \equiv 0$ in Eckart (1960, p. 60, equation 14) for the perturbations of a static equilibrium of a compressible fluid in a gravitational field. We also note that if we take the limit $c_0 \to \infty$ but *not* $\partial \rho_0/\partial r \to 0$, $\partial \rho_0/\partial z \to 0$, i.e. consider an incompressible fluid which is stratified, we find that Lord Rayleigh's criterion is relevant *only* when there is an axial force field. When there is no force field, the relevant stability criterion is easily deduced from theorem 1. In fact, when $V \equiv 0$ and $v_0 \neq 0$ almost everywhere, this theorem gives in the limit $c_0 \to \infty$ the following conditions

$$r^{-1}v_0^2\rho_0^{-1}\frac{\partial \rho_0}{\partial r} > 0 \quad \text{when} \quad r^{-3}\frac{\partial}{\partial r}(rv_0)^2 \geqslant 0, \tag{7.8}$$

$$r^{-3}\rho_0^{-1}\frac{\partial}{\partial r}(\rho_0 r^2 v_0^2) = r^{-1}v_0^2\rho_0^{-1}\frac{\partial \rho_0}{\partial r}+r^{-3}\frac{\partial}{\partial r}(rv_0)^2 > 0 \quad \text{when} \quad r^{-3}\frac{\partial}{\partial r}(rv_0)^2 < 0. \tag{7.9}$$

When $r^{-3}\partial(rv_0)^2/\partial r \geqslant 0$ the first inequality in (7.9) is an obvious consequence of (7.8), and when $r^{-3}\partial(rv_0)^2/\partial r < 0$ the first inequality in (7.8) is an obvious consequence of (7.9). Thus we have shown that in order for a stratified incompressible fluid to be stable when $V \equiv 0$ and $v_0 \neq 0$ almost everywhere, it is necessary that

$$\frac{\partial \rho_0}{\partial r} > 0 \quad \text{and} \quad \frac{\partial}{\partial r}(\rho_0 r^2 v_0^2) > 0 \tag{7.10}$$

are satisfied almost everywhere in the fluid. These are exactly the conditions given by Yih (1965, p. 271).

In order to interpret the general condition (7.3) further, we note that (2.4) implies

$$\begin{aligned}\frac{\alpha_1}{\alpha_2}r^{-1}v_0 b &= -\frac{1}{\alpha_2\beta_2}\{2(r^{-1}v_0 b)^2-\alpha_2\beta_1 r^{-1}v_0 b\} \\ &= -\frac{2}{\alpha_2\beta_2}\{(r^{-1}v_0 b-\tfrac{1}{4}\alpha_2\beta_1)^2-(\tfrac{1}{4}\alpha_2\beta_1)^2\}.\end{aligned} \tag{7.11}$$

Thus, when the necessary condition for stability (6.9) is satisfied, (7.11) shows that the right-hand side in the necessary condition (7.3) is negative if and only if one of the following two conditions is satisfied

$$\text{(i) } 0 < r^{-1}\frac{\partial}{\partial z}v_0^2 < \alpha_2\beta_1, \quad \text{(ii) } \alpha_2\beta_1 < r^{-1}\frac{\partial}{\partial z}v_0^2 < 0, \tag{7.12}$$

where by (2.8)

$$\alpha_2\beta_1 = -\left(\rho_0^{-1}\frac{\partial\rho_0}{\partial z} - c_0^{-2}\frac{\partial V}{\partial z}\right)\left(r^{-1}v_0^2 + \frac{\partial V}{\partial r}\right). \tag{7.13}$$

Furthermore, (7.11) shows that the right-hand side in (7.3) has a *minimum* value with respect to variations of the axial shear when

$$r^{-1}\frac{\partial}{\partial z}v_0^2 = \tfrac{1}{2}\alpha_2\beta_1. \tag{7.14}$$

For this value of the axial shear, (7.3) becomes

$$r^{-3}\frac{\partial}{\partial r}(rv_0)^2 > \frac{1}{4}\frac{\alpha_2}{\beta_2}(\beta_1)^2 = -\frac{1}{4}\left\{\rho_0^{-1}\frac{\partial\rho_0}{\partial z}\left(\frac{\partial V}{\partial z}\right)^{-1} - c_0^{-2}\right\}\left(r^{-1}v_0^2 + \frac{\partial V}{\partial r}\right)^2. \tag{7.15}$$

Thus we have shown that if the axial shear satisfies one of the two conditions in (7.12), the necessary condition for stability (7.3) is *less restrictive* with respect to the radial shear than (7.4). In fact, the condition (7.3) is somewhere between the two conditions (7.4) and (7.15). In this sense we see that an axial shear may have a *stabilizing effect* on the basic flow (2.2) in special cases. The magnitude of this stabilizing effect is by (7.15) seen to depend on the stratification of the fluid.

When the axial shear does not satisfy one of the two conditions (7.12), the necessary condition for stability (7.3) becomes *more restrictive* than (7.4) with respect to the radial shear. If the axial shear is very strong compared with the stratification of the fluid, (7.3) becomes very restrictive indeed. In fact, if

$$\left|r^{-1}\frac{\partial}{\partial z}v_0^2\right| \gg |\alpha_2\beta_1| \quad \text{and} \quad \left|r^{-1}\frac{\partial}{\partial z}v_0^2\right| \gg |\alpha_2\beta_2|, \tag{7.16}$$

then it follows from (7.3), (7.11) that a necessary condition for stability of the basic flow (2.2) is that

$$r^{-3}\frac{\partial}{\partial r}(rv_0)^2 \gg \left|r^{-1}\frac{\partial}{\partial z}v_0^2\right|. \tag{7.17}$$

We have shown therefore that an axial shear in most cases has a *destabilizing effect* on the basic flow (2.2), and that the magnitude of this destabilizing effect may become essential if the axial shear is sufficiently strong.

To illustrate the above results, let us again consider the classical problem of a flow between two coaxial rotating cylinders placed vertically in a gravitational field (with the z axis pointing upwards say, i.e. $\partial V/\partial z < 0$). When (6.9) is satisfied, $\alpha_2\beta_1 > 0$ by (7.13). Thus a sufficiently weak axial shear with $\partial(v_0^2)/\partial z > 0$ may have a stabilizing effect on the basic flow (2.2), while all axial shears with $\partial(v_0^2)/\partial z < 0$ and also all sufficiently strong axial shears with $\partial(v_0^2)/\partial z > 0$ have a destabilizing effect. In experiments where the length of the rotating cylinders has to be finite, this destabilizing effect may be important at least near the ends of the cylinders.

The results in this paper are also directly applicable to the basic motion of planetary and stellar atmospheres. In fact, $\partial V/\partial z \neq 0$ almost everywhere since $\partial V/\partial z = 0$ only in the equatorial plane $z = 0$ when the z axis is the axis of rotation and $z = r = 0$ is the centre of the planet or star. In the case of a star, the potential for the external forces V may here be due completely to the gravitational field set up by the unperturbed 'atmosphere' itself. The necessary conditions for stability of a rotating atmosphere obtained from theorem 2 are (6.9) and (7.3). As discussed earlier in this section, (6.9) implies that the atmosphere must be stratified in a (statically) stable way, while (7.3) implies that the outer regions of the atmosphere must rotate sufficiently fast compared with the regions near the axis (see Chandrasekhar 1961 for a discussion of the condition (7.4)).

Finally we should like to remark that it follows from the calculations in this paper, that in a stable basic flow (2.2) where $\partial V/\partial z \neq 0$ almost everywhere, the gravity waves will locally always oscillate with a frequency lying somewhere between two specific frequencies F_1 and F_2 which from (2.4), (2.8) and (3.5) are found to be

$$F_j^2 = \kappa_j = \tfrac{1}{2}E^2 + (-1)^j \left\{ \tfrac{1}{4}E^4 + 2r^{-1} v_0 \frac{\partial V}{\partial z} P \right\}^{\frac{1}{2}}, \quad j = 1, 2, \tag{7.18}$$

where
$$E^2 = N^2 + r^{-3} \frac{\partial}{\partial r}(rv_0)^2, \tag{7.19}$$

$$\begin{aligned} P &= c_0^{-1}\{\alpha_1 b + \alpha_2(a - 2r^{-1}v_0)\} \\ &= \left\{\rho_0^{-1}\frac{\partial \rho_0}{\partial r} - c_0^{-2}\left(r^{-1}v_0^2 + \frac{\partial V}{\partial r}\right)\right\}\frac{\partial v_0}{\partial z} - \left(\rho_0^{-1}\frac{\partial \rho_0}{\partial z} - c_0^{-2}\frac{\partial V}{\partial z}\right) r^{-1}\frac{\partial}{\partial r}(rv_0). \end{aligned} \tag{7.20}$$

Here N is the local Brunt–Väisälä frequency given by (7.1), (7.2). In the static case where $v_0 \equiv 0$ as well as in the limit $r \to \infty$, we see that

$$F_1 = 0, \quad F_2 = N. \tag{7.21}$$

Thus it seems that F_2 rather than N is the adequate generalization of the local Brunt–Väisälä frequency for the general basic states we have considered in this paper.

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